ZERO-HOPF POLYNOMIAL CENTERS OF THIRD-ORDER DIFFERENTIAL EQUATIONS

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Abstract. We study the 3-dimensional center problem at the zero-Hopf singularity in some families of polynomial vector fields arising from third-order polynomial differential equations. After proving some general properties we check that the quadratic family has no 3-dimensional centers. Later we characterize all the 3-dimensional centers in the cubic homogeneous family. Finally we give a partial classification of the 3-dimensional centers at just one singularity of the full cubic family and propose one open problem to close this classification.

1. Introduction and statement of the main results

The Kukles systems are described by second-order ordinary differential equations $\ddot{x} = g_3(x, \dot{x})$ where $g_3 \in \mathbb{R}_3[x, \dot{x}]$ is a real cubic polynomial. They were named like this since they were first studied by the Russian mathematician I. S. Kukles in [5]. The over dote denotes, as usual, derivative with respect to the time independent variable t. When $g_3(x, y) = -x + \cdots$ in [5] it is investigated the center problem at the monodromic singularity $(x, y) = (0, 0)$ of its associated planar vector field $\dot{x} = y$, $\dot{y} = g_3(x, y)$. Due to both theoretical and practical applications several papers have been published on the center problem for this kind of cubic systems, see for example [2, 10]. We recall that the characterization of the centers in analytic families of planar vector fields began with the pioneering works of Poincaré [8] and Liapunov [6]. See also the the book [9] for a modern approach based on computational algebra techniques.

In this paper we consider a more general situation, by considering higher-order Kukles systems. Indeed, we consider third-order ordinary differential equations

$$\dddot{x} = f_n(x, \dot{x}, \ddot{x}) \tag{1}$$

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with $f_n \in \mathbb{R}_n[x, \dot{x}, \ddot{x}]$, a real polynomial in three variables of degree n.

We transform the differential equation (1) in the usual way as a polynomial differential system

$$\dot{x} = y, \ \dot{y} = z, \ \dot{z} = f_n(x, y, z). \tag{2}$$

A real singularity $(x, y, z) = (x_0, 0, 0) \in \mathbb{R}^3$ of (2) is called a *zero-Hopf* singularity if its associated eigenvalues are $\{\pm i, 0\}$ with $i^2 = -1$. We say that a zero-Hopf singularity is a *3-dimensional center* if there is a neighborhood of it in $\mathbb{R}^3$ completely foliated by periodic orbits of (2), including continua of equilibria as trivial periodic orbits. As far as we know the only work that completely addresses this issue is [4] although in [7] it also appears in the context of the complete analytic local integrability at the zero-Hopf singularity.

The main results of this paper are the following. First we see how a discrete symmetry acts on any zero-Hopf singularity of (2) producing a 3-dimensional center.

Theorem 1. *If $f_n(x, -y, z) = -f_n(x, y, z)$ then any zero-Hopf singularity of (2) is a 3-dimensional center.*

The next result shows one reduction from the 3-dimensional center problem to the classical nondegenerate center problem of a planar vector field.

Theorem 2. *Let the origin be a zero-Hopf singularity of (2) with $\frac{\partial f_n}{\partial x} \equiv 0$. If the planar system $\dot{y} = z$, $\dot{z} = f_n(y, z)$ has a center at the origin in $\mathbb{R}^2$ then the origin in $\mathbb{R}^3$ is a 3-dimensional center of (2).*

Next result tell us that the minimum degree n needed for the appearance of 3-dimensional centers in family (2) is $n = 3$.

Theorem 3. *The quadratic ($n = 2$) vector field (2) has no 3-dimensional centers.*

Now we present the classification of all the 3-dimensional centers of family (2) when $n = 3$ and f_3 is a cubic homogeneous polynomial.

Theorem 4. *The cubic vector field (2) with $f_3(x, y, z)$ a cubic homogeneous polynomial satisfies the following:*

(i) *If $f_3(x, 0, 0) \not\equiv 0$ then there is no zero-Hopf singularity in the whole family (2).*

(ii) *When $f_3(x, 0, 0) \equiv 0$ family (2) has a 3-dimensional center if and only if it has the form*

$$\dot{x} = y, \ \dot{y} = z, \ \dot{z} = -K^2 x^2 y + A y^3 + B x y z + C y z^2 \tag{3}$$

with $K \neq 0$. Actually it has a continuum of 3-dimensional centers at any point of the x-axis except the origin.

When $n = 3$ and the origin is a zero-Hopf singularity of (2) with spectrum $\{\pm i\omega, 0\}$ and $\omega \neq 0$, the cubic polynomial f_3 is written as $f_3(x, y, z) = -\omega^2 y + \hat{f}_3(x, y, z)$ with $\hat{f}_3$ a polynomial of degree 3 without constant and linear terms. Performing a linear change of variables and a time rescaling $t \mapsto t/\omega$ the linear part of system (2) is written in real Jordan canonical form and (2) becomes

$$\dot{x} = -y + \omega^2 \mathcal{F}(x, y, z), \ \ \dot{y} = x, \ \ \dot{z} = \mathcal{F}(x, y, z) \tag{4}$$

where $\omega^3 \mathcal{F}(x, y, z) = \hat{f}_3(-x/\omega^2 + z, y/\omega, x)$.

Now we give a *partial* classification of the 3-dimensional centers at the origin of the full cubic ($n = 3$) family (2).

Theorem 5. *The cubic* ($n = 3$) *family* (2) *has a 3-dimensional center at the origin when the function* $\mathcal{F}(x, y, z)$ *of its associated family* (4) *is of the form:*

(i) $\omega^2 \mathcal{F}(x, y, z) = Fx^2 + Cxy - Fy^2 + Gx^2 z + Dxyz - Gy^2 z$ *with* $DF - CG = 0$*;*
(ii) $\omega^2 \mathcal{F}(x, y, z) = Cxy + Hx^2 y - \frac{H}{3} y^3 - \frac{4}{3} H\omega^2 xyz$*;*
(iii) $\omega^2 \mathcal{F}(x, y, z) = Hx^2 y - \frac{H}{3} y^3 + Gx^2 z + Dxyz - Gy^2 z$ *with* $GH = 0$*;*
(iv) $\omega^2 \mathcal{F}(x, y, z) = Cxy + Hx^2 y - Hy^3 - H\omega^2 xyz$ *with* $H \neq 0$*;*
(v) $\omega^2 \mathcal{F}(x, y, z) = Cxy + Hx^2 y - \frac{1}{2} H\omega^2 xyz$ *with* $H \neq 0$*;*
(vi) $\omega^2 \mathcal{F}(x, y, z) = \left(\frac{2D}{\omega^2} + H\right) y^3 + Hx^2 y + Dxyz$*;*
(vii) $\omega^2 \mathcal{F}(x, y, z) = Cxy + \frac{A}{\omega^2} yz + Hx^2 y + \frac{B}{\omega^4} y^3 + Dxyz + \frac{E}{\omega^2} yz^2$.

Moreover if the origin is a 3-dimensional center then $\mathcal{F}(x, y, z)$ *is either of the above forms or has the expressions:*

(viii) $\omega^2 \mathcal{F}(x, y, z) = Fx^2 + Cxy - Fy^2 + Ix^3 + Hx^2 y - 3Ixy^2 - \frac{H}{3} y^3 - \frac{3}{2} I\omega^2 x^2 z - H\omega^2 xyz + \frac{3}{2} I\omega^2 y^2 z$ *with* $2FH - 3CI = 0$*;*
(ix) $\omega^2 \mathcal{F}(x, y, z) = Fx^2 - \frac{DF}{IK^2} xy - Fy^2 + Ix^3 + Hx^2 y - 3Ixy^2 + \left(\frac{2D}{\omega^2} + H\right) y^3 - I\omega^2 x^2 z + Dxyz + I\omega^2 y^2 z$ *with* $I \neq 0$ *and* $2D^2 + 3DHK^2 + H^2 K^4 - I^2 K^4 = 0$*;*
(x) $\omega^2 \mathcal{F}(x, y, z) = Fx^2 + Cxy - Fy^2 + Hx^2 y - \frac{AF}{\omega^4} xy^2 + \frac{A}{\omega^2} yz - H\omega^2 xyz + \frac{AF}{\omega^2} y^2 z$*;*

Remark 6. We have proved that family (4) with the $\mathcal{F}(x, y, z)$ given in the statement (viii) of Theorem 5 has a 3-dimensional center at the origin in the parameter case $I = 0$. Otherwise if $I \neq 0$ then we have also showed that after a translation in the z-axis we can assume the parameters $F = C = 0$ in the expression of such a $\mathcal{F}$ without loss of generality.

On the other hand we have also checked that the origin of (4) with the $\mathcal{F}(x, y, z)$ stated in part (x) of Theorem 5 is a 3-dimensional center when the parameters are either $F = 0$ or $A = H = 0$.

All together Theorem 5 and Remark 6 implies that we will have the *complete* classification of the 3-dimensional centers at the origin in the cubic family (2) if and only if we can solve the next problem.

Open Problem *Find out if the origin is a 3-dimensional center for family* (4) *where* $\mathcal{F}(x, y, z)$ *is given by one of the following tree cases of Theorem 5: (viii) with* $I \neq 0$ *and* $F = C = 0$*; (ix); (x) with either* $F \neq 0$ *or* $A^2 + H^2 \neq 0$*.*

In analogy with the classical two-dimensional center problem in the qualitative theory of differential equations, the necessary 3-dimensional center conditions in the parameter space for family (4) are derived by vanishing the initial coefficients δ_j of the series of an adequate Poincaré map, see the next section. But the sufficient 3-dimensional center conditions (which guarantee that actually $\delta_j \equiv 0$ for any $j \in \mathbb{N}$) in the forthcoming proof of Theorem 5 needs the use of symmetry-integrability arguments. The technical difficulties why the cases stated in the above open problem cannot be solved are that, although we have checked that $\delta_j \equiv 0$ for $1 \leq j \leq 11$ so that it is very probable that the singularity is a 3-dimensional center, we have not been successful in finding the necessary symmetries or first integrals that ensure the existence of that 3-dimensional center.

2. The background and some auxiliary results

2.1. **The Poincaré map at the zero-Hopf singularity.** In this subsection we present a brief description of the theory introduced in [4] and next developed in [3]. There, it is considered an analytic three-dimensional system

$$\begin{aligned} \dot{x} &= -y + F_1(x, y, z) \\ \dot{y} &= x + F_2(x, y, z) \\ \dot{z} &= F_3(x, y, z), \end{aligned} \tag{5}$$

defined on a neighborhood $\mathcal{U} \subset \mathbb{R}^3$ of the origin and having a zero-Hopf singularity at the origin. Then, the F_i are real analytic functions on $\mathcal{U}$ without independent and linear terms.

Theorem 7 ([4, 3])**.** *We consider system* (5) *defined on a neighborhood* $\mathcal{U} \subset \mathbb{R}^3$ *of the origin. Let* $\delta > 0$ *be sufficiently large but fixed and define* $\mathcal{C}_\delta = \{(x, y, z) \in \mathcal{U} : z^2 > \delta(x^2 + y^2)\}$*, a thin solid cone with vertex at the origin surrounding the z-axis. Doing first the rescaling* $(x, y, z) \mapsto (x/\varepsilon, y/\varepsilon, z/\varepsilon)$ *and later the polar blow–up* $(x, y, z) \mapsto (\theta, r, w)$ *defined* *by*

$$x = r\cos\theta\ ,\ y = r\sin\theta\ ,\ z = rw\ , \tag{6}$$

system (5) *can be written in* $\mathcal{U}\backslash\mathcal{C}_\delta$*, for* $|r|$ *and* $|\varepsilon|$ *sufficiently small, as the analytic system*

$$\frac{dr}{d\theta} = \varepsilon R(\theta, r, w; \varepsilon)\ ,\quad \frac{dw}{d\theta} = \varepsilon W(\theta, r, w; \varepsilon)\ , \tag{7}$$

with invariant set $\{r = 0\}$ *and defined on the cylinder* $\{(\theta, r, w) \in \mathbb{S}^1 \times \mathbb{R} \times \mathcal{K}\}$ *where* $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ *and* $\mathcal{K} = \{w \in \mathbb{R} : |w| \leq \delta^2\}$.

Denoting by $\Psi(\theta; r_0, w_0; \varepsilon) = (r(\theta; r_0, w_0; \varepsilon), w(\theta; r_0, w_0; \varepsilon))$ the solution of (7) satisfying the initial condition $\Psi(0; r_0, w_0; \varepsilon) = (r_0, w_0) \in \mathbb{R} \times \mathcal{K}$ for $|r_0|$ sufficiently small, one can define the Poincaré translation map $\Pi(r_0, w_0; \varepsilon) = \Psi(2\pi; r_0, w_0; \varepsilon)$ and the analytic *displacement map* $d(r_0, w_0; \varepsilon) = \Pi(r_0, w_0; \varepsilon) - (r_0, w_0)$.

Remark 8. The need to restrict the values of w to the arbitrary but fixed compact set $\mathcal{K}$ containing the origin is clarified in [3]. From the geometry associated to the polar blow–up (6), we see that $(x, y, z) \in \mathcal{U}\backslash\mathcal{C}_\delta$ when $w \in \mathcal{K}$ and that (6) is a diffeomorphism in $\mathcal{U}\backslash\mathcal{C}_\delta$. Although the polar blow-up (6) does not cover $\mathcal{U}$, we emphasize that any periodic orbit of (5) in $\mathcal{U}$ not intersecting the z-axis is contained in $\mathcal{U}\backslash\mathcal{C}_\delta$ for δ sufficiently large. In consequence, the zeros of the displacement map $d(r_0, w_0)$, with $w_0 \in \mathcal{K}$ and $|r_0| \ll 1$, pick up all these periodic orbits. Anyway, see the arguments of [3] based on the properties real analytic functions of several variables that vanish on a set of positive measure, in [3] it is proved that the origin of system (5) is a 3-dimensional center if and only if $d(r_0, w_0; \varepsilon) \equiv 0$.

The idea behind the rescaling $(x, y, z) \mapsto (x/\varepsilon, y/\varepsilon, z/\varepsilon)$ in Theorem 7 is that now we can compute the Taylor series about $\varepsilon = 0$

$$d(r_0, w_0; \varepsilon) = \sum_{j\geq 1} \delta_j(r_0, w_0)\varepsilon^j$$

such that condition $d(r_0, w_0; \varepsilon) \equiv 0$ is equivalent to the vanish of all the coefficients $\delta_j(r_0, w_0)$. Using the terminology of the analogous two-dimensional problems in the qualitative theory of differential equations (see for example [1]), we will referer to the analytic function $\delta_j : \mathbb{R}^2 \to \mathbb{R}^2$ as the *j-th Melnikov function*.

The computation of the Melnikov functions δ_j is algorithmic although aid of an algebraic manipulator is highly recommended because the calculations involved are massive. First we expand in power series of ε both system (7) and its solution. Then we have

$$\begin{aligned} \frac{dr}{d\theta} &= \varepsilon R(\theta, r, w; \varepsilon) = \varepsilon \sum_{j\geq 1} R_j(\theta; r_0, w_0)\varepsilon^j \ , \\ \frac{dw}{d\theta} &= \varepsilon W(\theta, r, w; \varepsilon) = \varepsilon \sum_{j\geq 1} W_j(\theta; r_0, w_0)\varepsilon^j \ , \end{aligned} \tag{8}$$

and

$$\Psi(\theta; r_0, w_0; \varepsilon) = (r_0, w_0) + \left(\sum_{j\geq 1} \Psi_{1,j}(\theta; r_0, w_0)\varepsilon^j, \sum_{j\geq 1} \Psi_{2,j}(\theta; r_0, w_0)\varepsilon^j \right).$$

Since Ψ satisfies the initial condition $\Psi(0; r_0, w_0; \varepsilon) = (r_0, w_0)$ it follows that $\Psi_{i,j}(0; r_0, w_0) = 0$ for all $j \in \mathbb{N}$ and $i \in \{1, 2\}$. In summary the displacement map is

$$\begin{aligned} d(r_0, w_0; \varepsilon) &= \Psi(2\pi; r_0, w_0; \varepsilon) - (r_0, w_0) \\ &= \sum_{j\geq 1} \left(\Psi_{1,j}(2\pi; r_0, w_0), \Psi_{2,j}(2\pi; r_0, w_0)\right) \varepsilon^j, \end{aligned}$$

and consequently the j-th Melnikov function is

$$d_j(r_0, w_0) = \left(\Psi_{1,j}(2\pi; r_0, w_0), \Psi_{2,j}(2\pi; r_0, w_0)\right).$$

2.2. **Generalized reversibility and local analytic first integrals near equilibriums.** In this subsection we summarize some results obtained in [7] that we will need later in the proof of Theorem 1. The framework is to detect analytic first integrals near equilibrium points (located at the origin) of analytic vector fields

$$\dot{\mathbf{x}} = f(\mathbf{x}) = A\mathbf{x} + \cdots \tag{9}$$

in some open neighborhood of the origin in $\mathbb{C}^n$. The main argument is the well-known fact that any formal first integral of a (formal) Poincaré-Dulac normal form

$$\dot{\mathbf{x}} = \hat{f}(\mathbf{x}) = A\mathbf{x} + \cdots \tag{10}$$

of (9) is also a first integral of the linear system

$$\dot{\mathbf{x}} = A_s\mathbf{x} \tag{11}$$

where A_s is the semisimple part of the linearization A of (9). Recall that we can always decompose $A = A_s + A_n$ with A_S semisimple (diagonalizable in $\mathbb{C}$) and A_n is nilpotent with commutation $[A_s, A_n] = 0$. Also we remind that the formal vector field (10) is in Poincaré-Dulac normal form if $[A_s, \hat{f}] = 0$.

In this context a very interesting case to investigate is whether *all* formal first integrals of A_s are conserved by a normal form because this property may ensure convergence of the normal form and also complete local analytic integrability. The following result was first proven by Zhang [11, 12], see also Theorem 9 in [7].

Theorem 9 ([11, 12] and [7]). *Assume that $A_s \neq 0$ and the linear system* (11) *in $\mathbb{C}^n$ admits $n-1$ independent polynomial first integrals. If some formal Poincaré-Dulac normal form of* (9) *admits $n-1$ independent formal first integrals, then* (9) *admits a convergent transformation to Poincaré-Dulac normal form and also it possesses $n-1$ independent analytic first integrals.*

Now we adapt the general Proposition 11 of [7] to our purpose, hence we state it in the very particular case that we need. We say that (9) is *time-reversible* if there is a nonsingular $n \times n$ matrix T such that $T^{-1} \circ f \circ T = -f$.

Proposition 10 ([7]). *Consider system* (9) *with $n = 3$ and diagonal linear part $A = A_s$ with spectrum $\{\pm i\omega, 0\}$ where $\omega \in \mathbb{R}\backslash\{0\}$. If* (9) *is time-reversible then the two independent first integrals of* (11) *are conserved in a Poincaré-Dulac normal form of* (9).

3. Proof of Theorem 1

First we observe that any real singularity $(x_0, 0, 0)$ of (2) can be placed at the origin via the translation $(x, y, z) \mapsto (x - x_0, y, z)$ without affecting the form of (2). Hence without loss of generality we will assume that the origin is a zero-Hopf singularity of (2). In particular $f_n(x, y, z) = -\omega^2 y + \hat{f}_n(x, y, z)$ with $\omega \neq 0$ and $\hat{f}_n$ a polynomial of degree n without constant nor linear terms such that the eigenvalues of the linearization at the origin of (2) are $\{\pm i\omega, 0\}$. Moreover we can perform a linear change of variables $(x, y, z)^T \mapsto P^{-1}(x, y, z)^T$ where

$$P = \begin{pmatrix} -\frac{1}{\omega^2} & 0 & 1 \\ 0 & \frac{1}{\omega} & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and a time rescaling $t \mapsto t/\omega$ such that the linear part of system (2) is written in real Jordan canonical form. In short, after all these transformations (2) becomes

$$\begin{aligned} \dot{x} &= -y + \omega^2 \mathcal{F}(x, y, z) \\ \dot{y} &= \;\; x \\ \dot{z} &= \mathcal{F}(x, y, z) \end{aligned} \tag{12}$$

where $\mathcal{F}(x, y, z) = \frac{1}{\omega^3} \hat{f}_n(-x/\omega^2 + z, y/\omega, x)$.

We also can transform system (12) via the linear complex change of variables $(x,y,z) \mapsto (X,Y,Z) = (x+iy, x-iy, z)$ diagonalizing in $\mathbb{C}$ the linear part of (12). In this way system (12) becomes a system in $\mathbb{C}^3$ with the form

$$
\begin{aligned}
\dot{X} &= iX + \omega^2 \mathcal{F}\left(\tfrac{X+Y}{2}, \tfrac{i(Y-X)}{2}, Z\right), \\
\dot{Y} &= -iY + \omega^2 \mathcal{F}\left(\tfrac{X+Y}{2}, \tfrac{i(Y-X)}{2}, Z\right), \\
\dot{Z} &= \mathcal{F}\left(\tfrac{X+Y}{2}, \tfrac{i(Y-X)}{2}, Z\right).
\end{aligned} \tag{13}
$$

It is straightforward to check that this complex system is time-reversible with respect to the linear involution $(X,Y,Z) \mapsto (Y,X,Z)$ that exchanges X and Y if and only if $\mathcal{F}$ is an odd function in second variable, i.e., $\mathcal{F}(x,-y,z) = -\mathcal{F}(x,y,z)$. Notice that this symmetry is inherited by f_n, hence $f_n(x,-y,z) = -f_n(x,y,z)$. In this symmetric case both first integrals of the linear part are conserved in a Poincaré-Dulac normal form of (13), see Proposition 10. Hence the complex system admits two independent analytic first integrals $\hat{H}_1(X,Y,Z) = X^2 - Y^2 + \cdots$ and $\hat{H}_2(X,Y,Z) = Z + \cdots$, see Theorem 9, which implies that the real system (12) possesses two independent locally real analytic first integrals $H_1(x,y,z) = x^2 + y^2 + \cdots$ and $H_2(x,y,z) = z + \cdots$. Then the origin of (12) is a 3-dimensional center and consequently the singularity $(x_0, 0, 0)$ of (2) is too.

4. Proof of Theorem 2

We take f_n independent of x so that in system (2) a decoupling occurs. Thus we deal with the planar subsystem

$$
\dot{y} = z\ ,\ \ \dot{z} = f_n(y,z) = -\omega^2 y + \cdots \tag{14}
$$

where the starting terms in $f_n(y,z)$ are derived from the assumption of having a zero-Hopf singularity of (2) at the origin, see the beginning of the proof of Theorem 1. By hypothesis (14) has a nondegenerate center at the origin. Therefore the solution $\xi(t;(y_0,z_0)) = (y(t;(y_0,z_0)), z(t;(y_0,z_0)))$ of (14) passing at time $t=0$ through the point $(y_0,z_0) \in U \subset \mathbb{R}^2$ in a sufficiently small neighborhood U of the origin is $T(y_0,z_0)$-periodic for any $(y_0,z_0) \in U$ and also there is an analytic first integral $H(y,z) = y^2 + z^2 + \cdots$ defined in U of (14) . Going back to the full system (2) in $\mathbb{R}^3$, it has the solution $(x(t;(x_0,y_0,z_0)), \xi(t;(y_0,z_0)))$ with initial condition $(x_0,y_0,z_0) \in \mathbb{R} \times U$ and where $x(t;(x_0,y_0,z_0)) = x_0 + \int_0^t y(t;(y_0,z_0))\, dt$. We claim that actually the function $x(t;(x_0,y_0,z_0))$ is also a $T(y_0,z_0)$-periodic function implying that the origin of (2) is a 3-dimensional center. To prove the claim we only need to prove that the function $y(t;(y_0,z_0))$ has zero average in the interval $[0,T(y_0,z_0)]$. This is true because this average is

$$\int_0^{T(y_0,y_0)} y(t;(y_0,z_0))\,dt = \oint_{H=h} dx \equiv 0 \text{ for all } (y_0,z_0)\in U,$$

where in the last step we integrate an exact 1-form over closed level curves $\{H(y,z)=h\}$ where $h=H(y_0,z_0)$.

Remark 11. We observe Theorem 2 can be restated in terms of the transformed system (12) as follows. If $\frac{\partial \mathcal{F}}{\partial z}\equiv 0$ and the planar vector field $\dot{x}=-y+\omega^2\mathcal{F}(x,y)$, $\dot{y}=x$ has a center at the origin in $\mathbb{R}^2$ then the origin in $\mathbb{R}^3$ is a 3-dimensional center of (12).

5. Proof of Theorem 3

We take $f_2(x,y,z)=a_0(y,z)+a_1(y,z)x+a_2(y,z)x^2$. A singularity $p_0=(x_0,0,0)\in\mathbb{R}^3$ of (2) corresponds to a real solution of the quadratic equation $f_2(x_0,0,0)=0$. It is easy to check that when $a_2(0,0)=0$ then either there are no singularities of (2) when $a_1(0,0)=0$ or there is just one singularity but without any zero associated eigenvalue otherwise. So we continue the proof assuming that $a_2(0,0)\neq 0$. We impose the condition

$$a_0(0,0)=\frac{a_1(0,0)^2}{4a_2(0,0)} \tag{15}$$

in order to have a null eigenvalue. Next we have the conditions

$$\frac{\partial a_0}{\partial z}(0,0) = \frac{a_1(0,0)\left[2a_2(0,0)\frac{\partial a_1}{\partial z}(0,0)-a_1(0,0)\frac{\partial a_2}{\partial z}(0,0)\right]}{4a_2^2(0,0)}, \tag{16}$$

$$\begin{aligned} 0 > {} & 4a_2^2(0,0)\frac{\partial a_0}{\partial y}(0,0)-2a_1(0,0)a_2(0,0)\frac{\partial a_1}{\partial y}(0,0)+ \\ & a_1^2(0,0)\frac{\partial a_2}{\partial y}(0,0) \end{aligned} \tag{17}$$

for having also two non-zero pure imaginary eigenvalues.

Since f_2 is a polynomial of degree 2 we have

$$\begin{aligned} a_0(y,z) &= b_0+b_1y+b_2z+b_3y^2+b_4yz+b_5z^2, \\ a_1(y,z) &= c_0+c_1y+c_2z, \\ a_2(y,z) &= d_0. \end{aligned}$$

Then $a_2(0,0)\neq 0$, (15) and (16) produces

$$d_0\neq 0,\ b_0=\frac{c_0^2}{4d_0},\ b_2=\frac{c_0c_2}{2d_0},\ b_1=\frac{2c_0c_1d_0-K^2}{4d_0^2}$$

where in the last condition we have introduced a new parameter $K \neq 0$ such that inequality (17) reads for $0 > -K^2$. The coordinates of the zero-Hopf singularity are $p_0 = (-c_0/(2d_0), 0, 0)$ whose eigenvalues are $\{\pm i\omega, 0\}$ with $\omega = K/(2d_0)$. As in the first paragraph of the proof of Theorem 1, we translate p_0 to the origin; we perform a linear transformation to get the real Jordan canonical form of the linear part of the system and we rescale the time $t \mapsto t/\omega$. Thus we obtain that (2) is transformed into a system (12) of the form

$$
\begin{aligned}
\dot{x} &= -y + \omega^2 \mathcal{F}(x, y, z) \\
\dot{y} &= \ \ x \\
\dot{z} &= \mathcal{F}(x, y, z)
\end{aligned}
\tag{18}
$$

with

$$
\begin{aligned}
\mathcal{F}(x, y, z) \ = \ & \frac{4Cd_0^2}{K^2}x^2 + \frac{4Bd_0^2}{K^2}xy + \frac{32b_3d_0^5}{K^5}y^2 + \frac{4Ad_0^2}{K^2}xz + \\
& \frac{16c_1d_0^4}{K^4}yz + \frac{8d_0^4}{K^3}z^2
\end{aligned}
$$

after the reparametrization $(c_2, b_4, b_5) \mapsto (A, B, C)$ with

$$
c_2 = \frac{16d_0^4 + AK^3}{2d_0K^2}, \ b_4 = \frac{16c_1d_0^4 + BK^4}{4d_0^2K^2}, \ b_5 = \frac{32d_0^6 + 4Ad_0^2K^3 + CK^5}{2d_0K^4}.
$$

Using the theory developed in [4] and summarized in subsection 2.1 gives that family (18) with $d_0 \neq 0$ and $K \neq 0$ has not a 3-dimensional center at the origin because the first associated Melnikov function $\delta_1(r_0, w_0)$ is

$$
\delta_1(r_0, w_0) = \left(A\pi r_0^2 w_0, -\frac{\pi r_0}{K^5}[-32b_3d_0^5 - 4Cd_0^2K^3 - 16d_0^4K^2w_0^2 + AK^5w_0^2] \right)
$$

and clearly $\delta_1(r_0, w_0) \not\equiv 0$ for all possible choice of the parameters. Hence the theorem follows.

6. Proof of Theorem 4

If $n = 3$ and f_3 is a cubic homogeneous polynomial, then is a simple exercise to check that there is no zero-Hopf singularity in family (2) if $f_3(x, 0, 0) \not\equiv 0$. Hence statement (i) is straightforward to check and we concentrate our attention only in proving statement (ii).

Following the notation of the proof of Theorem 3 and taking into account that $f_3(x, 0, 0) \equiv 0$ we take $f_3(x, y, z) = a_0(y, z) + a_1(y, z)x +$

$a_2(y,z)x^2$ with

$$\begin{aligned} a_0(y,z) &= b_6y^3 + b_7y^2z + b_8yz^2 + b_9z^3, \\ a_1(y,z) &= c_3y^2 + c_4yz + c_5z^2, \\ a_2(y,z) &= d_1y + d_2z. \end{aligned}$$

Under these conditions it is easy to check that system (2) has zero-Hopf singularities only when $d_2 = 0$ and $d_1 = -K^2$ with $K \neq 0$. In short a continuum of singularities $(x_0, 0, 0)$ on the x-axis appear for any $x_0 \in \mathbb{R}$ such that all of them except the origin are zero-Hopf points with eigenvalues $\{\pm i\omega, 0\}$ where $\omega = Kx_0$.

Pick up one of these zero-Hopf singularities $(x_0, 0, 0)$ with $x_0 \neq 0$ and translate it to the origin by means of $(x, y, z) \mapsto (x - x_0, y, z)$. Next we follow the first paragraph of the proof of Theorem 1 and we do a linear change of variables to obtain the linear part of the system in real Jordan canonical form and also we perform the time rescaling $t \mapsto t/\omega$. After all these transformations system (2) is written as a system (12) of the form

$$\begin{aligned} \dot{x} &= -y + \omega^2 \mathcal{F}(x,y,z) \\ \dot{y} &= \ \ x \\ \dot{z} &= \mathcal{F}(x,y,z) \end{aligned} \tag{19}$$

where $\mathcal{F}$ is the following cubic polynomial

$$\begin{aligned} \mathcal{F}(x,y,z) = & -\frac{c_5}{K^3x_0^2}x^2 - \frac{2 + c_4x_0^2}{K^4x_0^5}xy - \frac{c_3}{K^5x_0^4}y^2 + \frac{2}{K^2x_0^3}yz + \\ & \frac{b_9K^2x_0^2 - c_5}{K^5x_0^5}x^3 + \frac{-1 - c_4x_0^2 + b_8K^2x_0^4}{K^6x_0^8}x^2y - \\ & \frac{c_3 - b_7K^2x_0^2}{K^7x_0^7}xy^2 + \frac{b_6}{K^6x_0^6}y^3 + \frac{c_5}{K^3x_0^3}x^2z + \frac{2 + c_4x_0^2}{K^4x_0^6}xyz + \\ & \frac{c_3}{K^5x_0^5}y^2z - \frac{1}{K^2x_0^4}yz^2. \end{aligned}$$

Using the theory developed in [4] and summarized in subsection 2.1 gives that the first associated Melnikov functions $\delta_i(r_0, w_0)$ at the origin for family (19) are the following.

$$\delta_1(r_0, w_0) = \left(0, -\frac{\pi r_0(c_3 + c_5K^2x_0^2)}{K^5x_0^4}\right)$$

and clearly $\delta_1(r_0, w_0) \equiv 0$ if and only if $c_3 = -c_5K^2x_0^2$. Next we compute

$$\delta_2(r_0, w_0) = \left(\frac{\pi r_0^3(b_7 - 4c_5 + 3b_9K^2x_0^2)}{4K^3x_0^3}, -\frac{\pi r_0^2 w_0(b_7 - 12c_5 + 3b_9K^2x_0^2}{4K^3x_0^3}\right)$$

which vanishes identically only when $c_5 = 0$ and $b_7 = -3b_9K^2x_0^2$. After we get that

$$\delta_3(r_0, w_0) = \left(-\frac{3b_9\pi r_0^4 w_0}{2Kx_0^2}, \frac{b_9\pi r_0^3(2 + c_4x_0^2 + 3K^4w_0^2x_0^4)}{2K^5x_0^6}\right)$$

and $\delta_3(r_0, w_0) \equiv 0$ if and only if $b_9 = 0$. Also we can check that $\delta_4(r_0, w_0) = \delta_5(r_0, w_0) \equiv 0$.

The resulting family (2) becomes (3) after the relabeling of parameters $(b_6, c_4, b_8) = (A, B, C)$. The proof finishes noticing that in (3) we have that $f_3(x, y, z) = -K^2x^2y + Ay^3 + Bxyz + Cyz^2$ is an odd function in the variable y and applying Theorem 1.

7. Proof of Theorem 5

Let $f_3(x, y, z) = a_0(y, z) + a_1(y, z)x + a_2(y, z)x^2 + a_3x^3$. The singularity $p_0 = (x_0, 0, 0) \in \mathbb{R}^3$ of (2) corresponds to a real solution of the cubic equation $f_3(x_0, 0, 0) = 0$. Since we want the origin $(0, 0, 0)$ to be a singular point of (2) with $n = 3$ we must have $a_0(0, 0) = 0$. Then we get $a_1(0, 0) = 0$ to have a null eigenvalue associated to the origin and moreover

$$\frac{\partial a_0}{\partial z}(0, 0) = 0, \quad \frac{\partial a_0}{\partial y}(0, 0) < 0, \tag{20}$$

for having also two non-zero pure imaginary eigenvalues.

Now we take the following expression of the polynomial f_3:

$$\begin{aligned} a_0(y, z) &= b_0 + b_1y + b_2z + b_3y^2 + b_4yz + b_5z^2 + b_6y^3 + \\ &\quad b_7y^2z + b_8yz^2 + b_9z^3, \\ a_1(y, z) &= c_0 + c_1y + c_2z + c_3y^2 + c_4yz + c_5z^2, \\ a_2(y, z) &= d_0 + d_1y + d_2z, \\ a_3 &= e_0. \end{aligned}$$

Conditions $a_0(0, 0) = a_1(0, 0) = 0$ and (20) yield

$$b_0 = c_0 = b_2 = 0, \;\; b_1 = -K^2$$

with $K \neq 0$. The resulting system (2) with $n = 3$ has a singularity at the origin with eigenvalues $\{\pm i\omega, 0\}$ with $\omega = K$.

As in the first paragraph of the proof of Theorem 1, after performing a linear transformation to get the real Jordan canonical form of the linear part of the system and doing a time rescaling $t \mapsto t/\omega$ we obtain that (2) is transformed into a system (12) of the form

$$\begin{aligned} \dot{x} &= -y + K^2\mathcal{F}(x,y,z) \\ \dot{y} &= \quad x \\ \dot{z} &= \mathcal{F}(x,y,z) \end{aligned} \tag{21}$$

with

$$\begin{aligned} \mathcal{F}(x,y,z) \;=\; & \frac{F}{K^2}x^2 + \frac{b_3}{K^5}y^2 + \frac{A}{K^2}xz + \frac{c_1}{K^4}yz + \frac{d_0}{K^3}z^2 + \frac{I}{K^2}x^3 \\ & + \frac{C}{K^2}xy + \frac{H}{K^2}x^2y + \frac{E}{K^2}xy^2 + \frac{b_6}{K^6}y^3 + \frac{G}{K^2}x^2z + \\ & \frac{D}{K^2}xyz + \frac{c_3}{K^5}y^2z + \frac{B}{K^2}xz^2 + \frac{d_1}{K^4}yz^2 + \frac{e_0}{K^3}z^3 \end{aligned}$$

after the reparametrization

$$(c_2, d_2, b_4, c_4, b_7, b_5, b_8, b_9) \mapsto (A, B, C, D, E, F, G, H, I)$$

with

$$\begin{aligned} c_2 &= \frac{2d_0 + AK^3}{K^2},\ d_2 = \frac{3e_0 + BK^3}{K^2},\ b_4 = \frac{c_1 + CK^4}{K^2}, \\ c_4 &= \frac{2d_1 + DK^4}{K^2},\ b_7 = \frac{c_3 + EK^5}{K^2},\ b_5 = \frac{d_0 + AK^3 + FK^5}{K^4}, \\ c_5 &= \frac{3e_0 + 2BK^3 + GK^5}{K^4},\ b_8 = \frac{d_1 + DK^4 + HK^6}{K^4}, \\ b_9 &= \frac{e_0 + BK^3 + GK^5 + IK^7}{K^6}. \end{aligned}$$

Now we compute Melnikov functions (as explained in subsection 2.1) for family (21) with $K \neq 0$ in order to detect 3-dimensional centers at the origin. The first associated Melnikov function $\delta_1(r_0, w_0)$ is

$$\delta_1(r_0, w_0) = \Big(A\pi r_0^2 w_0, -\frac{\pi r_0}{K^5}[-b_3 - FK^3 + K^2(-2d_0 + AK^3)w_0^2]\Big)$$

Therefore $\delta_1(r_0, w_0) \equiv 0$ if and only if

$$A = d_0 = 0,\ b_3 = -FK^3. \tag{22}$$

Under these constrains, the second Melnikov function $\delta_2(r_0, w_0) = (\delta_{2,1}(r_0, w_0), \delta_{2,2}(r_0, w_0))$ has components

$$\begin{aligned} \delta_{2,1}(r_0, w_0) &= -\frac{\pi}{4K^4} r_0^3 [c_1 F + EK^4 + 3IK^4 + 4BK^4 w_0^2], \\ \delta_{2,2}(r_0, w_0) &= -\frac{\pi}{4K^5} r_0^2 w_0 [-4c_3 + 5c_1 FK - 4GK^3 + EK^5 + 3IK^5 \\ & \quad -8e_0 K^2 w_0^2 + 4BK^5 w_0^2]. \end{aligned}$$

Next it is easy to check that the second Melnikov function $\delta_2(r_0, w_0)$ vanishes identically if and only if

$$
\begin{array}{rcl}
B & = & e_0 = 0, \\
c_3 & = & \dfrac{1}{4}(5c_1FK - 4GK^3 + EK^5 + 3IK^5), \\
E & = & -\dfrac{1}{K^4}(c_1F + 3IK^4).
\end{array} \tag{23}
$$

Straightforward computations gives the following higher order Melnikov functions. More precisely we get

$$
\delta_3(r_0, w_0) = \left(-\Delta_1 \frac{\pi}{4K^4} r_0^4 w_0, \frac{\pi}{4K^6} r_0^3[\Delta_2 - 3K^2\Delta_3 w_0^2]\right)
$$

where Δ_i are polynomials in the parameter space of family (21) whose expressions can be found in the Appendix 1. Also we obtain $\delta_4(r_0, w_0) = (\delta_{4,1}(r_0, w_0), \delta_{4,2}(r_0, w_0))$ with components

$$
\begin{array}{rcl}
\delta_{4,1}(r_0, w_0) & = & -\dfrac{\pi}{48K^8} r_0^5[\Delta_4 + 4\Delta_1(3c_1 + CK^4)w_0 + 6K^2\Delta_5 w_0^2], \\
\delta_{4,2}(r_0, w_0) & = & \dfrac{\pi}{48K^{10}} r_0^4[-4(3c_1 - 4CK^4)\Delta_2 + K^2\Delta_6 w_0 + \\
& & 4K^2\Delta_7 w_0^2 + 6K^4\Delta_8 w_0^3].
\end{array}
$$

The fifth-order Melnikov function $\delta_5(r_0, w_0) = (\delta_{5,1}(r_0, w_0), \delta_{5,2}(r_0, w_0))$ has components

$$
\begin{array}{rcl}
\delta_{5,1}(r_0, w_0) & = & -\dfrac{\pi}{576K^{12}} r_0^6[12\Delta_9 - \Delta_{10}w_0 + 24K^2\Delta_{11}w_0^2 + 18K^4\Delta_{12}w_0^3], \\
\delta_{5,2}(r_0, w_0) & = & \dfrac{\pi}{48K^{14}} r_0^5[\Delta_{13} + 12K^2\Delta_{14}w_0 - K^2\Delta_{15}w_0^2 + 24K^4\Delta_{16}w_0^3 + \\
& & 18K^6\Delta_{17}w_0^4].
\end{array}
$$

The Bautin ideal $\mathcal{B} =< \Delta_i : i \in \mathbb{N} >$ at the origin of (21) is generated by all the *Poincaré-Liapunov constants* Δ_i and is a polynomial ideal in the polynomial ring $\mathbb{R}[\lambda]$ where

$$
\lambda = (K, C, D, F, G, H, I, b_6, c_1, d_1) \in \mathbb{R}^{10}
$$

contains some of the parameters of family (21). Recall that several parameters have been fixed, see (22) and (23).

Since $\mathcal{B}$ is Noetherian it is generated by a finite number of polynomials by the Hilbert basis theorem. But unfortunately we do not know this basis a priori. Let $\mathcal{B}_k =< \Delta_1, \ldots, \Delta_k >$ be the ideal generated by the first k Poincaré-Liapunov constants at the origin of (21). Before computing the *center variety* $\mathbf{V}(\mathcal{B})$ we will do some simplifications. We define $\hat{\mathcal{B}}_k =< \hat{\Delta}_1, \ldots, \hat{\Delta}_k >$ be the ideal generated by some polynomials $\hat{\Delta}_i$ that we define sequentially as follows. Let $\hat{\Delta}_1 = \Delta_1$ and

$\hat{\Delta}_i \equiv \Delta_i \mod \hat{\mathcal{B}}_{i-1}$ for $i \geq 2$. Clearly $\hat{\mathcal{B}}_k = \mathcal{B}_k$. In the former reduction we obtain that $\hat{\Delta}_7 = \hat{\Delta}_{11} = \hat{\Delta}_{12} = \hat{\Delta}_{16} = \hat{\Delta}_{17} = 0$. Relabeling consecutively the subscripts of the $\hat{\Delta}_i$ such that we remove the above null polynomials we can work with the ideal $\hat{\mathcal{B}}_{11} =< \hat{\Delta}_1, \ldots, \hat{\Delta}_{11} >$. We expect that $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\hat{\mathcal{B}}_{11})$ where the inclusion $\mathbf{V}(\mathcal{B}) \subset \mathbf{V}(\hat{\mathcal{B}}_{11})$ is obvious. To verify the opposite inclusion we will find the minimal irreducible decomposition of the variety $\mathbf{V}(\hat{\mathcal{B}}_{11})$. This decomposition will be performed with the help of a computer algebra system, more precisely with the procedure `minAssGTZ` from the library `primdec.lib` of the software Singular using the degree-reverse lexicographic order with $K > C > D > F > G > H > I > b_6 > c_1 > d_1$ to find the primary decomposition of $\sqrt{\hat{\mathcal{B}}_{11}}$, the radical of $\hat{\mathcal{B}}_{11}$. The output is that $\mathbf{V}(\hat{\mathcal{B}}_{11}) = \cup_{i=1}^{13} \mathbf{V}(J_i)$ where the irreducible varieties $\mathbf{V}(J_i)$ are the varieties associated to the ideals J_i listed in Appendix 2.

Finally we must verify the each point of the varieties $\mathbf{V}(J_i)$ corresponds with a system (21) having a 3-dimensional center at the origin. We shall see that sometimes this is not true and we need to compute higher order Melnikov functions $\delta_j(r_0, w_0)$ with $j \geq 6$ at the origin of family (21) restricted to $\lambda \in \mathbf{V}(J_i)$.

Now we recall that in family (21) the parameter $K \neq 0$ and therefore we do not take into account any of the varieties $\mathbf{V}(J_i)$ with $i \in \{2, 11, 12\}$ because they have the polynomial K as generator.

7.1. **The variety $\mathbf{V}(J_1)$.** Since the variety $\mathbf{V}(J_1)$ is defined by

$$\mathbf{V}(J_1) = \{\lambda \in \mathbb{R}^{10} : d_1 = c_1 = b_6 = I = H = DF - CG = 0\}$$

we get the associated family (21) with

$$K^2 \mathcal{F}(x, y, z) = Fx^2 + Cxy - Fy^2 + Gx^2 z + Dxyz - Gy^2 z \tag{24}$$

and the condition $DF - CG = 0$.

By Theorem 1 and Remark 11 we have that if $F = G = 0$ or $D = G = 0$ then (21) has a 3-dimensional center at the origin.

Therefore only remains the study of the case $G \neq 0$. In this case we can put $C = DF/G$ and after rescaling the time by $t \mapsto G\,t$ system (21) becomes

$$\begin{aligned} \dot{x} &= -Gy + K^2 \hat{\mathcal{F}}(x, y, z) \\ \dot{y} &= \ \ Gx \\ \dot{z} &= \hat{\mathcal{F}}(x, y, z), \end{aligned} \tag{25}$$

with $K^2 \hat{\mathcal{F}}(x, y, z) = (Gx^2 + Dxy - Gy^2)(F + Gz)$. Performing the planar rotation in the (x, y)-plane of angle $\theta^* = -\frac{1}{2}\mathrm{arccot}\,(D/(2G))$

system (25) is greatly simplified because $2\cos(2\theta^*) + (D\sin(2\theta^*)/G = 0$. Indeed, doing the change of variables

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \cos\theta^* & -\sin\theta^* & 0 \\ \sin\theta^* & \cos\theta^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{26}$$

system (25) becomes

$$\begin{aligned} \dot{u} &= -v + C_1\mathcal{G}(u,v,w), \\ \dot{v} &= \quad u + C_2\mathcal{G}(u,v,w), \\ \dot{w} &= C_3\mathcal{G}(u,v,w), \end{aligned} \tag{27}$$

where $\mathcal{G}(u,v,w) = uv(F+Gw)$ and C_i are certain constants depending on the parameters of the family. Removing common factors we see that two planar subsystems are associated to (27), namely

$$\dot{u} = -1 + C_1 u(F+Gw), \ \dot{w} = C_3 u(F+Gw), \tag{28}$$

and

$$\dot{v} = 1 + C_2 v(F+Gw), \ \dot{w} = C_3 v(F+Gw). \tag{29}$$

Since the origin is a regular point of both (28) and (29) there are around it analytic first integrals $\hat{H}_1(u,w)$ of (28) and $\hat{H}_2(v,w)$ of (29). Additionally both functions $\hat{H}_1(u,w)$ and $\hat{H}_2(v,w)$ are first integrals of the full system (27). Let $\nabla = (\partial_u, \partial_v, \partial_w)$ be the gradient operator. Since the gradient vectors $\nabla\hat{H}_1(u,w) = (\partial_u\hat{H}_1, 0, \partial_w\hat{H}_1)$ and $\nabla\hat{H}_2(v,w) = (0, \partial_v\hat{H}_2, \partial_w\hat{H}_2)$ are linearly independent except in the zero Lebesgue measure set $\{(u,v,w) : \partial_u\hat{H}_1 = \partial_v\hat{H}_2 = 0\}$ we conclude that $\hat{H}_1(u,w)$ and $\hat{H}_2(v,w)$ are two functionally independent analytical first integrals of (27) almost everywhere in a neighborhood of the origin, that is in a full Lebesgue measure (dense) neighborhood of the origin. In short, going back through the linear change (26) one has that the pull back $H_1(x,y,z) = \hat{H}_1(\cos\theta^* x - \sin\theta^* y, z)$ and $H_2(x,y,z) = \hat{H}_2(\sin\theta^* x + \cos\theta^* y, z)$ also are functionally independent analytical first integrals of family (21) with $\mathcal{F}$ given by (24). Taking into account that the linear part of (21) there is no restriction in assuming that $H_1(x,y,z) = x^2 + y^2 + \cdots$ and $H_2(x,y,z) = z + \cdots$ and therefore family (21) with $\mathcal{F}$ given by (24) has a center at the origin. This proves statement (i) of Theorem 5.

7.2. **The variety $\mathbf{V}(J_3)$.** Taking into account the expressions of the generators of the ideal J_3 we see that the variety $\mathbf{V}(J_3)$ is given by

$$\mathbf{V}(J_3) = \{\lambda \in \mathbb{R}^{10} : d_1 = c_1 = H = I = G = D = b_6 = 0\}$$

and therefore $\mathbf{V}(J_3) \subset \mathbf{V}(J_1)$.

7.3. **The variety $\mathbf{V}(J_4)$.** From the generators of the ideal J_4 we get that

$$\begin{aligned}\mathbf{V}(J_4) &= \{\lambda \in \mathbb{R}^{10} : d_1 = c_1 = F = 0, G = -2K^2 I,\\ &\quad D = -\frac{4}{3}K^2 H, b_6 = \frac{1}{4}K^2 D\}.\end{aligned}$$

Imposing that $\lambda \in \mathbf{V}(J_4)$ we can compute the sixth-order Melnikov function at the origin of (21)

$$\delta_6(r_0, w_0) = I(H^2 + 9I^2)\pi \left(\frac{5}{864}r_0^7, \frac{7}{288}r_0^6 w_0\right)$$

from where we see that $\delta_6(r_0, w_0) \equiv 0$ if and only $I = 0$. Therefore its associated family (21) has

$$K^2\mathcal{F}(x,y,z) = Cxy + Hx^2y - \frac{H}{3}y^3 - \frac{4}{3}HK^2xyz, \tag{30}$$

and by Theorem 1 family (21) has a 3-dimensional center at the origin. This proves statement (ii) of Theorem 5.

7.4. **The variety $\mathbf{V}(J_5)$.** From the generators of the ideal J_5 one has that

$$\begin{aligned}\mathbf{V}(J_5) &= \{\lambda \in \mathbb{R}^{10} : d_1 = c_1 = 0, G = -\frac{3}{2}K^2 I, D = -K^2 H,\\ &\quad b_6 = \frac{1}{3}K^2 D, 2FH - 3CI = 0\}\end{aligned}$$

hence we get the associated family (21) with

$$\begin{aligned}K^2\mathcal{F}(x,y,z) &= Fx^2 + Cxy - Fy^2 + Ix^3 + Hx^2y - 3Ixy^2 - \frac{H}{3}y^3 -\\ &\quad \frac{3}{2}IK^2x^2z - HK^2xyz + \frac{3}{2}IK^2y^2z,\end{aligned} \tag{31}$$

and the condition $2FH - 3CI = 0$. Then statement (viii) of Theorem 5 follows.

By Theorem 1 and Remark 11 we have that if $F = I = 0$ or $H = I = 0$ then (21) has a 3-dimensional center at the origin.

Hence only remains the study of $I \neq 0$ and we take $C = 2FH/(3I)$. Doing the z-translation $(x, y, z) \mapsto (x, y, z - \alpha)$ with $\alpha = 2F/(3IK^2)$ we obtain the same system (21) with $K^2\mathcal{F}(x, y, z)$ given by (31) but with $F = C = 0$. Under these new constrains one has $\delta_i(r_0, w_0) \equiv 0$ for all $i \leq 11$. This proves Remark 6.

7.5. **The variety $\mathbf{V}(J_6)$.** From the generators of the ideal J_6 we get

$$\begin{aligned}\mathbf{V}(J_5) &= \{\lambda \in \mathbb{R}^{10} : d_1 = c_1 = F = C = 0, b_6 = -\frac{1}{3}K^4 H,\\ &\quad 2GH - 3DI = 0\}.\end{aligned}$$

If $\lambda \in \mathbf{V}(J_6)$ we known that the first Melnikov function $\delta_i(r_0, w_0) \equiv 0$ for $i = 1, \ldots, 5$ but $\delta_6(r_0, w_0) \not\equiv 0$. After some involved computations we can see that $\delta_6(r_0, w_0) \equiv 0$ if and only if the following polynomials in $\mathbb{R}[\lambda]$

$$\begin{aligned}\Lambda_1 &= 8DGH - 6D^2 I + 24G^2 I + 10GH^2K^2 - 6DHIK^2 +\\ &\quad 54GI^2K^2 + 3H^2IK^4 + 27I^3K^4,\\ \Lambda_2 &= 32DGH - 30D^2 I + 72G^2 I + 66GH^2K^2 - 78DHIK^2 +\\ &\quad 126GI^2K^2 + 3H^2IK^4 + 27I^3K^4\end{aligned}$$

vanishes. If we define $\Lambda_3 = 2GH - 3DI$, one of the generators of the ideal J_5, and compute resultants between the Λ_i with respect to G yields

$$\begin{aligned}\mathcal{R}[\Lambda_1, \Lambda_3, G] &= 12I(H^2 + 9I^2)(D + HK^2)(2D + HK^2),\\ \mathcal{R}[\Lambda_2, \Lambda_3, G] &= 12I(H^2 + 9I^2)(D + HK^2)(6D + HK^2).\end{aligned}$$

Therefore two cases arise to annul the above resultants:

(a) Put the parameter $I = 0$. Then the associated family (21) has

$$K^2\mathcal{F}(x, y, z) = Hx^2y - \frac{H}{3}y^3 + Gx^2z + Dxyz - Gy^2z, \tag{32}$$

with the condition $GH = 0$. Clearly if $G = 0$ then by Theorem 1 family (21) has a 3-dimensional center at the origin, hence we can assume that $H = 0$ and $G \neq 0$. In this last case, family (21) has $\mathcal{F}(x, y, z)$ as in (24) (with the particular election of parameters $F = C = 0$) and we have already proved that family (21) has a 3-dimensional center at the origin. Then statement (iii) of Theorem 5 is proved.

(b) Let $D = -HK^2$ and $I \neq 0$. Then $\delta_i(r_0, w_0) \equiv 0$ for $i = 6, 7, 8$ if and only if $G = -3IK^2/2$, in which case family (21) has

$$\begin{aligned}K^2\mathcal{F}(x, y, z) &= Ix^3 + Hx^2y - 3Ixy^2 - \frac{H}{3}y^3 - \frac{3}{2}IK^2x^2z - HK^2xyz +\\ &\quad \frac{3}{2}IK^2y^2z \end{aligned}\tag{33}$$

with $I \neq 0$. In this case, the associated family (21) has $\mathcal{F}(x, y, z)$ as in (31) (with the parameters $F = C = 0$) and we have already proved that in this case family (21) has a 3-dimensional center at the origin.

7.6. **The variety $\mathbf{V}(J_7)$.** From the generators of the ideal J_7 we obtain that

$$\mathbf{V}(J_5) = \{\lambda \in \mathbb{R}^{10} : d_1 = c_1 = H = I = b_6 = C = F = D = G = 0\}$$

giving rise to the trivial linear system $\dot{x} = -y$, $\dot{y} = x$, $\dot{z} = 0$.

7.7. **The variety $\mathbf{V}(J_8)$.** From the generators of the ideal J_8 we see that in order to express the variety $\mathbf{V}(J_8)$ we must impose the following parameter constraints

$$d_1 = c_1 = 0, G = -K^2 I, b_6 = K^4 H + 2K^2 D$$

but additional polynomial restrictions are needed to determine the variety $\mathbf{V}(J_8)$. Then we will split the problem in several subcases.

Fist we assume that $I = F = H = 0$. Then we obtain also $D = 0$ and $\lambda \in \mathbf{V}(J_8)$ but the resulting associated family (21) is a particular case of system (24).

Assume now that $I = F = 0$ but $H \neq 0$ and $D = -HK^2$. Then $\lambda \in \mathbf{V}(J_8)$ and the associated family (21) has

$$K^2 \mathcal{F}(x, y, z) = Cxy + Hx^2 y - Hy^3 - HK^2 xyz, \tag{34}$$

with $H \neq 0$. By Theorem 1 family (21) has a 3-dimensional center at the origin. This proves statement (iv) of Theorem 5.

We can suppose now that $I = F = 0$, $H \neq 0$ and $D \neq -HK^2$. Then we necessarily get $D = -HK^2/2$ to have $\lambda \in \mathbf{V}(J_8)$. In this case family (21) has

$$K^2 \mathcal{F}(x, y, z) = Cxy + Hx^2 y - \frac{1}{2} HK^2 xyz, \tag{35}$$

with $H \neq 0$. By Theorem 1 family (21) has a 3-dimensional center at the origin and statement (v) of Theorem 5 is proved.

Let $I = 0$ but $F \neq 0$. If $D \neq 0$ then $\lambda \notin \mathbf{V}(J_8)$, so we assume that $D = 0$. Then $\lambda \in \mathbf{V}(J_8)$ if and only if $H = 0$. Hence family (21) becomes a particular case of system (24).

Finally we deal with the case $I \neq 0$. Then we have $C = -DF/(IK^2)$ and $\lambda \in \mathbf{V}(J_8)$ if and only if $2D^2 + 3DHK^2 + H^2K^4 - I^2K^4 = 0$. We also check that under this constrained one has $\delta_i(r_0, w_0) \equiv 0$ for $i = 1, \ldots, 8$. The resulting family (21) has

$$\begin{aligned} K^2 \mathcal{F}(x, y, z) &= Fx^2 - \frac{DF}{IK^2} xy - Fy^2 + Ix^3 + Hx^2 y - 3Ixy^2 + \\ &\quad \left(\frac{2D}{K^2} + H\right) y^3 - IK^2 x^2 z + Dxyz + IK^2 y^2 z, \end{aligned} \tag{36}$$

with the conditions $I \neq 0$ and $2D^2 + 3DHK^2 + H^2K^4 - I^2K^4 = 0$. Then statement (ix) of Theorem 5 is proved. Notice that Theorem 1 and Remark 11 do not work in this case because $I \neq 0$. Also we have seen that, in the special case $F = 0$ then $\delta_i(r_0, w_0) \equiv 0$ for $i = 1, \dots, 11$.

7.8. **The variety $\mathbf{V}(J_9)$.** From the generators of the ideal J_9 we see that

$$\begin{aligned} \mathbf{V}(J_9) &= \{\lambda \in \mathbb{R}^{10} : d_1 = c_1 = F = C = 0, G = -K^2 I, \\ &\quad b_6 = K^4 H + 2K^2 D\} \end{aligned}$$

We get that either $I \neq 0$ and then the resulting family (21) becomes a particular case with $F = 0$ of (36) or $I = 0$ and therefore the associated family (21) has

$$K^2 \mathcal{F}(x, y, z) = \left(\frac{2D}{K^2} + H \right) y^3 + Hx^2 y + Dxyz. \tag{37}$$

Using Theorem 1 we obtain that (21) has a 3-dimensional center at the origin proving statement (vi) of Theorem 5.

7.9. **The variety $\mathbf{V}(J_{10})$.** The generators of the ideal J_{10} yield

$$\mathbf{V}(J_{10}) = \{\lambda \in \mathbb{R}^{10} : d_1 = b_6 = I = G = 0, D = -K^2 H\}.$$

Hence the associated family (21) has

$$\begin{aligned} K^2 \mathcal{F}(x, y, z) &= Fx^2 + Cxy - Fy^2 + Hx^2 y - \frac{c_1 F}{K^4} xy^2 + \frac{c_1}{K^2} yz - \\ &\quad HK^2 xyz + \frac{c_1 F}{K^2} y^2 z. \end{aligned} \tag{38}$$

Then statement (x) of Theorem 5 is proved changing the name $c_1 \mapsto A$. Further computations reveal that $\delta_i(r_0, w_0) \equiv 0$ for $i = 1, \dots, 10$. We notice that by Theorem 1 and Remark 11 we have that if $F = 0$ or $c_1 = H = 0$ then (21) has a 3-dimensional center at the origin proving thus the second part of Remark 6.

7.10. **The variety $\mathbf{V}(J_{13})$.** The generators of the ideal J_{13} yield

$$\mathbf{V}(J_{13}) = \{\lambda \in \mathbb{R}^{10} : I = G = F = 0\}.$$

Hence the associated family (21) has

$$\begin{aligned} K^2 \mathcal{F}(x, y, z) &= Cxy + \frac{c_1}{K^2} yz + Hx^2 y + \frac{b_6}{K^4} y^3 + Dxyz + \\ &\quad \frac{d_1}{K^2} yz^2. \end{aligned} \tag{39}$$

By Theorem 1 family (21) has a 3-dimensional center at the origin. This proves statement (vii) of Theorem 5 after changing the name of parameters $(c_1, b_6, d_1) \mapsto (A, B, E)$.

8. Appendix 1

$$
\begin{aligned}
\Delta_1 &= -2d_1F + c_1G + 3c_1IK^2,\\
\Delta_2 &= -b_6F + DFK^2 - CGK^2 + FHK^4 - 2CIK^4,\\
\Delta_3 &= 2d_1F + c_1G - c_1IK^2,\\
\Delta_4 &= -5b_6c_1F - 2c_1^2I - 6Cd_1FK^2 + 3c_1DFK^2 + 16b_6GK^2 +\\
&\quad 2c_1CGK^2 + 2b_6CFK^4 + 3c_1FHK^4 + 18b_6IK^4 + 3c_1CIK^4 -\\
&\quad 2CDFK^6 + 2C^2GK^6 + 8GHK^6 - 4DIK^6 - 2CFHK^8 +\\
&\quad 4C^2IK^8 + 6HIK^8,\\
\Delta_5 &= -13c_1d_1F - 4c_1^2G + 9c_1^2IK^2 + 2Cd_1FK^4 + 2c_1CGK^4 + 6d_1IK^4,\\
\Delta_6 &= -47b_6c_1F - 14c_1^2I - 42Cd_1FK^2 + 33c_1DFK^2 + 4b_6GK^2 -\\
&\quad 52c_1CGK^2 + 2b_6CFK^4 + 33c_1FHK^4 + 18b_6IK^4 -\\
&\quad 57c_1CIK^4 - 2CDFK^6 + 2C^2GK^6 + 20GHK^6 - 28DIK^6 -\\
&\quad 2CFHK^8 + 4C^2IK^8 + 6HIK^8,\\
\Delta_7 &= -24c_1d_1F - 6c_1^2G + 18c_1^2IK^2 - 10Cd_1FK^4 - 7c_1CGK^4 +\\
&\quad 3c_1CIK^6,\\
\Delta_8 &= -33c_1d_1F - 24c_1^2G - 8d_1GK^2 + 9c_1^2IK^2 + 2Cd_1FK^4 +\\
&\quad 2c_1CGK^4 + 6d_1IK^4,\\
\Delta_9 &= 4b_6c_1^2F - 16b_6d_1FK^2 - 8c_1Cd_1FK^2 - 4c_1^2DFK^2 + 8b_6c_1GK^2 +\\
&\quad 8c_1^2CGK^2 - 9b_6c_1CFK^4 - 4c_1^2FHK^4 + 24b_6c_1IK^4 +\\
&\quad 18c_1^2CIK^4 - 6C^2d_1FK^6 - 4b_6DFK^6 + 7c_1CDFK^6 -\\
&\quad 8d_1F^3K^6 + 16b_6CGK^6 - 2c_1C^2GK^6 + 4c_1F^2GK^6 -\\
&\quad 8d_1FHK^6 + 4c_1GHK^6 + 2b_6C^2FK^8 + 4D^2FK^8 - 4CDGK^8 +\\
&\quad 7c_1CFHK^8 + 18b_6CIK^8 - 5c_1C^2IK^8 + 12c_1F^2IK^8 +\\
&\quad 12c_1HIK^8 - 2C^2DFK^{10} + 2C^3GK^{10} + 4DFHK^{10} +\\
&\quad 8CGHK^{10} - 12CDIK^{10} - 2C^2FHK^{12} + 4C^3IK^{12} +\\
&\quad 6CHIK^{12},\\
\Delta_{10} &= 168c_1^2d_1F - 84c_1^3G + 294b_6c_1^2FK^2 - 108c_1^3IK^2 +\\
&\quad 540b_6d_1FK^4 + 810c_1Cd_1FK^4 - 150c_1^2DFK^4 - 246b_6c_1GK^4 +\\
&\quad 159c_1^2CGK^4 - 84b_6c_1CFK^6 + 48d_1DFK^6 + 24Cd_1GK^6 -\\
&\quad 36c_1DGK^6 - 150c_1^2FHK^6 - 810b_6c_1IK^6 - 381c_1^2CIK^6 +\\
&\quad 12C^2d_1FK^8 - 24b_6DFK^8 + 60c_1CDFK^8 + 208d_1F^3K^8 -\\
&\quad 24b_6CGK^8 - 150c_1C^2GK^8 + 40c_1F^2GK^8 + 84d_1FHK^8 -\\
&\quad 210c_1GHK^8 - 72Cd_1IK^8 + 84c_1DIK^8 + 24D^2FK^{10} -\\
&\quad 24CDGK^{10} + 60c_1CFHK^{10} - 246c_1C^2IK^{10} -\\
&\quad 168c_1F^2IK^{10} - 270c_1HIK^{10} + 24DFHK^{12} +\\
&\quad 24CGHK^{12} - 96CDIK^{12},
\end{aligned}
$$

$$\begin{aligned}
\Delta_{11} &= -78c_1^2d_1F - 12c_1^3G - 12d_1^2FK^2 + 6c_1d_1GK^2 + 66c_1^3IK^2 - \\
&\quad 13c_1Cd_1FK^4 + 2c_1^2CGK^4 + 54c_1d_1IK^4 - 12d_1DFK^6 - \\
&\quad 6c_1DGK^6 + 15c_1^2CIK^6 + 2C^2d_1FK^8 + 2c_1C^2GK^8 + \\
&\quad 6Cd_1IK^8 + 6c_1DIK^8, \\
\Delta_{12} &= -120c_1^2d_1F - 75c_1^3G - 64d_1^2FK^2 - 64c_1d_1GK^2 + 45c_1^3IK^2 + \\
&\quad 20c_1Cd_1FK^4 + 20c_1^2CGK^4 + 72c_1d_1IK^4 + 8d_1DFK^6 + \\
&\quad 8Cd_1GK^6 + 8c_1DGK^6, \\
\Delta_{13} &= 84b_6c_1^2F - 24c_1^3I + 240b_6d_1FK^2 + 48c_1Cd_1FK^2 - \\
&\quad 108c_1^2DFK^2 + 72b_6c_1GK^2 + 108c_1^2CGK^2 + 378b_6^2FK^4 - \\
&\quad 63b_6c_1CFK^4 - 108c_1^2FHK^4 - 72b_6c_1IK^4 + 204c_1^2CIK^4 + \\
&\quad 90C^2d_1FK^6 - 366b_6DFK^6 + 123c_1CDFK^6 + 72d_1F^3K^6 + \\
&\quad 294b_6CGK^6 - 93c_1C^2GK^6 - 24c_1F^2GK^6 + 48d_1FHK^6 + \\
&\quad 72c_1GHK^6 + 48Cd_1IK^6 - 72c_1DIK^6 + 210b_6C^2FK^8 + \\
&\quad 36D^2FK^8 + 200b_6F^3K^8 - 36CDGK^8 - 132b_6FHK^8 + \\
&\quad 123c_1CFHK^8 + 612b_6CIK^8 - 246c_1C^2IK^8 - 96c_1F^2IK^8 - \\
&\quad 24c_1HIK^8 - 210C^2DFK^{10} - 200DF^3K^{10} + 210C^3GK^{10} + \\
&\quad 200CF^2GK^{10} - 162DFHK^{10} + 90CGHK^{10} + 48CDIK^{10} - \\
&\quad 96FGIK^{10} - 210C^2FHK^{12} - 200F^3HK^{12} - 198FH^2K^{12} + \\
&\quad 420C^3IK^{12} + 400CF^2IK^{12} + 348CHIK^{12} - 144FI^2K^{12}, \\
\Delta_{14} &= 39b_6c_1^2F - 2c_1^3I - 104b_6d_1FK^2 - 70c_1Cd_1FK^2 - 41c_1^2DFK^2 - \\
&\quad 16b_6c_1GK^2 + 30c_1^2CGK^2 - 79b_6c_1CFK^4 - 24d_1DFK^4 + \\
&\quad 24Cd_1GK^4 - 41c_1^2FHK^4 + 114b_6c_1IK^4 + 121c_1^2CIK^4 - \\
&\quad 54C^2d_1FK^6 - 20b_6DFK^6 + 61c_1CDFK^6 - 64d_1F^3K^6 - \\
&\quad 88c_1C^2GK^6 - 16c_1F^2GK^6 - 88d_1FHK^6 - 8c_1GHK^6 + \\
&\quad 48Cd_1IK^6 - 4c_1DIK^6 + 2b_6C^2FK^8 + 20D^2FK^8 - \\
&\quad 20CDGK^8 + 61c_1CFHK^8 + 18b_6CIK^8 - 113c_1C^2IK^8 + \\
&\quad 48c_1F^2IK^8 + 54c_1HIK^8 - 2C^2DFK^{10} + 2C^3GK^{10} + \\
&\quad 20DFHK^{10} + 24CGHK^{10} - 76CDIK^{10} - 2C^2FHK^{12} + \\
&\quad 4C^3IK^{12} + 6CHIK^{12},
\end{aligned}$$

$$
\begin{aligned}
\Delta_{15} \;=\;& 816c_1^2d_1F+132c_1^3G+1428b_6c_1^2FK^2-288d_1^2FK^2+\\
&144c_1d_1GK^2-36c_1^3IK^2+2304b_6d_1FK^4+3648c_1Cd_1FK^4-\\
&780c_1^2DFK^4+1518b_6c_1GK^4+2979c_1^2CGK^4+432c_1d_1IK^4-\\
&84b_6c_1CFK^6-96d_1DFK^6+888Cd_1GK^6+36c_1DGK^6-\\
&780c_1^2FHK^6-810b_6c_1IK^6+735c_1^2CIK^6+312C^2d_1FK^8-\\
&24b_6DFK^8+60c_1CDFK^8+832d_1F^3K^8-24b_6CGK^8+\\
&150c_1C^2GK^8+664c_1F^2GK^8+192d_1FHK^8-102c_1GHK^8+\\
&648Cd_1IK^8+804c_1DIK^8+24D^2FK^{10}-24CDGK^{10}+\\
&60c_1CFHK^{10}-246c_1C^2IK^{10}-168c_1F^2IK^{10}-\\
&270c_1HIK^{10}+24DFHK^{12}+24CGHK^{12}-96CDIK^{12},\\
\Delta_{16} \;=\;& -233c_1^2d_1F-122c_1^3G-24d_1^2FK^2-36c_1d_1GK^2+\\
&111c_1^3IK^2-63c_1Cd_1FK^4-48c_1^2CGK^4+90c_1d_1IK^4-\\
&28d_1DFK^6-16Cd_1GK^6-22c_1DGK^6+15c_1^2CIK^6+\\
&2C^2d_1FK^8+2c_1C^2GK^8+6Cd_1IK^8+6c_1DIK^8,\\
\Delta_{17} \;=\;& -260c_1^2d_1F-215c_1^3G-144d_1^2FK^2-224c_1d_1GK^2+\\
&45c_1^3IK^2+20c_1Cd_1FK^4+20c_1^2CGK^4+72c_1d_1IK^4+\\
&8d_1DFK^6+8Cd_1GK^6+8c_1DGK^6.
\end{aligned}
$$

9. Appendix 2

$$
\begin{aligned}
J_1 &= < d_1, c_1, b_6, I, H, DF - CG >, \\
J_2 &= < d_1, c_1, b_6, K >, \\
J_3 &= < d_1, c_1, H^2 + 9I^2, 2GH - 3DI, \\
&\quad DH + 6GI, 3G^2 - 4Hb_6, DG + 8Ib_6, \\
&\quad 3D^2 + 16Hb_6, 2K^2I + G, 4K^2H + 3D, \\
&\quad 6K^2D - 4b_6 >, \\
J_4 &= < d_1, c_1, F, 2GH - 3DI, \\
&\quad DG + 8Ib_6, 3D^2 + 16Hb_6, 2K^2I + G, \\
&\quad 4K^2H + 3D, K^2D - 4b_6 >, \\
J_5 &= < d_1, c_1, 2GH - 3DI, 2FH - 3CI, \\
&\quad 2DG + 9Ib_6, DF - CG, D^2 + 3Hb_6, \\
&\quad 3K^2I + 2G, K^2H + D, 2CG^2 + 9FIb_6, \\
&\quad K^2D - 3b_6, K^2CG - 3Fb_6 >, \\
J_6 &= < d_1, c_1, F, C, 2GH - 3DI, \\
&\quad K^4H + 3b_6, K^4DI + 2Gb_6 >, \\
J_7 &= < d_1, c_1, H^2 + 9I^2, 2GH - 3DI, \\
&\quad 2FH - 3CI, DH + 6GI, CH + 6FI, \\
&\quad DF - CG, D^2 + 4G^2, CD + 4FG, \\
&\quad C^2 + 4F^2, K^4H + 3b_6, 3K^4I^2 - Hb_6, \\
&\quad 2K^4GI - Db_6, 2K^4FI - Cb_6, \\
&\quad K^4DI + 2Gb_6, K^4CI + 2Fb_6 >,
\end{aligned}
$$

$$
\begin{aligned}
J_8 &= < d_1, c_1, DF - CG, K^2 I + G, \\
&\quad G^2 H - 2DGI - I^2 b_6, \\
&\quad DGH - 2D^2 I + G^2 I - HIb_6, \\
&\quad G^3 - GHb_6 + DIb_6, FG^2 - FHb_6 + CIb_6, \\
&\quad 2DG^2 I - GH^2 b_6 + DHIb_6 + GI^2 b_6, \\
&\quad 2CG^2 I - FH^2 b_6 + CHIb_6 + FI^2 b_6, \\
&\quad FGH^2 - 3CGHI + 2CDI^2 - FGI^2, \\
&\quad F^2 H^2 - 3CFHI + 2C^2 I^2 - F^2 I^2, \\
&\quad K^2 GH + 2DG + Ib_6, K^2 DH + 2D^2 - G^2 + Hb_6, \\
&\quad 2D^2 G^2 - D^2 Hb_6 + 3DGIb_6 - H^2 b_6^2 + I^2 b_6^2, \\
&\quad K^2 G^2 - K^2 Hb_6 - Db_6, \\
&\quad K^2 H^2 b_6 + 2DG^2 + DHb_6 + GIb_6, \\
&\quad 4D^2 GI^2 - GH^3 b_6 + DH^2 Ib_6 + GHI^2 b_6 + 2DI^3 b_6, \\
&\quad 4CDGI^2 - FH^3 b_6 + CH^2 Ib_6 + FHI^2 b_6 + 2CI^3 b_6, \\
&\quad 4D^3 GI - D^2 H^2 b_6 + 8D^2 I^2 b_6 - 3G^2 I^2 b_6 - H^3 b_6^2 + 4HI^2 b_6^2, \\
&\quad K^2 FH^2 + 3CGH - 2CDI + FGI, K^4 H + 2K^2 D - b_6, \\
&\quad 8D^3 I^3 - GH^4 b_6 + DH^3 Ib_6 - GH^2 I^2 b_6 + 8DHI^3 b_6 + 2GI^4 b_6, \\
&\quad 8CD^2 I^3 - FH^4 b_6 + CH^3 Ib_6 - FH^2 I^2 b_6 + 8CHI^3 b_6 + 2FI^4 b_6, \\
&\quad 8D^4 I^2 - D^2 H^3 b_6 + 10D^2 HI^2 b_6 - H^4 b_6^2 + 2H^2 I^2 b_6^2 - I^4 b_6^2 >, \\
J_9 &= < d_1, c_1, F, C, K^2 I + G, G^2 H - 2DGI - I^2 b_6, \\
&\quad K^2 GH + 2DG + Ib_6, K^4 H + 2K^2 D - b_6 >, \\
J_{10} &= < d_1, b_6, I, G, K^2 H + D >, \\
J_{11} &= < d_1, b_6, I, G, K >, \\
J_{12} &= < c_1, F, K >, \\
J_{13} &= < I, G, F > .
\end{aligned}
$$

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